Low Correlation Zone Sequences over Finite Fields and Rings

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Sequences with low correlations are used as signature sequences in spread spectrum communication technologies.
Periodic Correlation Function

Definition

Given two complex sequences \( a = \{a_0, a_1, \ldots, a_{N-1}\} \) and \( b = \{b_0, b_1, \ldots, b_{N-1}\} \) of period \( N \), we define the periodic cross-correlation between \( a \) and \( b \) as

\[
R_{a,b}(\tau) = \sum_{i=0}^{N-1} a_i \, b^*_i \tau, \quad 0 \leq \tau < N,
\]

where \( \star \) denotes for complex conjugation.

- A key measure to determine similarity between two sequences in the CDMA context.
- Helps in determining optimal sequences for synchronization, code acquisition and channel identification.
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Low Correlation Zone Requirement

\[ |R_{a,b}(\tau)| \]

\( \tau = -N \) \hspace{2cm} \( \tau = 0 \)

\( a = b \) \hspace{2cm} \( \tau = N \)

General Asynchronous CDMA

\[ |R_{a,b}(\tau)| \]

\( \tau = -N \) \hspace{2cm} \( \tau = 0 \)

\( a = b \) \( l_{cz} \) \hspace{2cm} \( \tau = N \)

Quasi-Synchronous CDMA

Figure: Correlation Zone
Orthogonality

Consider,

\[ \mathcal{F} = \{ a_0^r, a_1^r, \ldots, a_{N-1}^r \}; r = 1 \cdots M \],

a family of \( M \) sequences of each length \( N \).

We say the family is **Orthogonal**, if:

\[ R_{r,s}(\tau) = \sum_{i=0}^{N-1} a_i^r a_{i+\tau}^{s*}, \begin{cases} = N, & \text{for } \tau = 0, r = s \\ = 0, & \text{for } \tau = 0, r \neq s \end{cases} \]

\( *: \) Complex Conjugate Note that \( R_{r,s}(\tau) \) is zero only for \( \tau = 0 \).

Walsh sequence set or vectors of Hadamard matrices satisfy the above criterion. They are widely used in synchronous CDMA communication systems.
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For a small number \( \epsilon (\epsilon < N) \) and a zone \( l_{cz} \), we say the family is **Generalized Quasi-Orthogonal** \((GQO(N, M, \epsilon, l_{cz})\), if :

\[
R_{r,s}(\tau) = \sum_{i=0}^{N-1} a_r^r a_{i+\tau}^{s*}, \begin{cases} 
= N, & \text{for } \tau = 0, r = s \\
= \epsilon, & \text{for } \tau = 0, r \neq s \\
= \epsilon, & 0 < |\tau| \leq l_{cz}
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\]

A requirement is that maximum non-trivial correlation amongst members of a \( GQO \) should be a small quantity.

Sequences satisfying this property \( l_{cz} = 0 \) are studied by [Yang, Kim, Kumar, 2000]: Quasi-orthogonal CDMA systems, IEEE-IT, 2000.
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Polyphase sequences

- The above definitions are for any complex valued sequences. In practice the signature sequences in CDMA are generally defined over points on unit circle.

- Given two $p$-ary sequences $a = \{a_0, a_1, \ldots, a_{N-1}\}$ and $b = \{b_0, b_1, \ldots, b_{N-1}\}$ of period $N$, we define the periodic cross-correlation between $a$ and $b$ as

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Aperiodic Correlation requirements

Consider $GQO(N, M, \epsilon, l_{cz})$. $M$ sequences of length $M$, with low correlation zone $l_{cz}$ and low correlation value $\epsilon$. We also sometimes need aperiodic correlations of sequences in $(GQO(N, M, \epsilon, l_{cz}))$ satisfy the following property:

$$
\delta_{r,s}(\tau) = \sum_{i=0}^{N-\tau} \omega^{a_r^i - a_s^{i+\tau}}, \begin{cases} 
= N, & \text{for } \tau = 0, r = s \\
\leq \epsilon, & \text{for } \tau = 0, r \neq s \\
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In general it is difficult to optimise both aperiodic and periodic correlations.

The usual plan is first construct sequences with optimal periodic properties and then choose appropriate sequences with smaller aperiodic correlations.
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Bounds on correlations for Generalized Quasi-Orthogonal sequences

- **Welch Bound on Correlations**: For a family $M$ periodic complex sequences of length $N$, the periodic correlations satisfy

$$R(\tau) \geq \sqrt{N}.$$ 

- **Sidelnikov Bound**: For the binary case,

$$R(\tau) \geq \sqrt{2N}.$$ 

Several variations of the above bounds exist. But they all assume that $l_{cz}$ is equal to length of the sequences. Hence they do not provide sufficient hint for optimal constructions.
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Welch Boound on Periodic Sequences

Consider set of $M$ vectors of length $N$.

$$\{(a^s_1, a^s_2, \cdots a^s_N), 0 \leq s < M\}, \text{ each of unit norm}$$

$$R_{s,s}(0) = \sum_{i=0}^{N-1} |a_i^s|^2 = 1.$$ 

Let $\delta$ be the non-trivial value of inner product between any two vectors in the matrix:

Inner Product $\delta(r, s) = R_{r,s}(0) = \sum_{i=0}^{N-1} a_i^r (a_i^s)^*$. 

$$\delta^{2k} = R_{s,t}(0)^{2k} \geq \frac{1}{(M-1)} \left[ \frac{M}{(N+k-1)} - 1 \right]$$
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Construct a new matrix $H$ from $M$ vectors of the previous slide: Include all vectors as rows of $H$ and further add $1, \cdots, l_{cz}$ shifts of these vectors to $H$.

$$H(iN + j, t) := a_{i+j}^t,$$

where $0 \leq i \leq M - 1$, $0 \leq j \leq l_{cz} - 1$ and $0 \leq t < N$. Clearly there are $Ml_{cz}$ vectors in $H$; Using Inner-product theorem

$$\delta^{2k} \geq \frac{1}{(Ml_{cz} - 1)} \left[ \frac{Ml_{cz}}{\binom{N+k-1}{k}} - 1 \right].$$

Because of $R_{s,t}(\tau) = R_{t,s}(-\tau)^*$,

$$R_{s,t}(\tau) \geq \frac{1}{(Ml_{cz} - 1)} \left[ \frac{Ml_{cz}}{\binom{N+k-1}{k}} - 1 \right], 0 \leq |\tau| \leq l_{cz}. $$
Bound for Generalized Quasi-Orthogonal Sequences

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Special Cases

- Welch’s result assumes that norm of the vectors is unity, i.e.,
  \[ R_{r,r}(0) = \sum_{i=0}^{N-1} |a_r^i|^2 = 1, \ 0 \leq r < M. \]

- \( \delta^2 \) is the value of non-trivial correlation between any two sequences with shifts in the range \( 0 \leq \tau < l_{cz} \).

In practice, the sequences are over unit circle (polyphase or biphase), then \( R_{r,r}(0) = N. \)

- Choose \( k = 1 \) in the previous result and apply the previous theorem by normalizing sequences, we get
  \[ \delta^2 \geq N \frac{(Ml_{cz} - N)}{Ml_{cz} - 1}. \]

- Rearranging, we get
  \[ (Ml_{cz} - 1)(1 - \delta^2/N) \leq N - 1. \]
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Zero/low correlation zone constraints

Zero Correlation Zone If $\delta^2 / N = 0$, then we have,

$$Ml_{cz} \leq N.$$ 

If $l_{cz} = 1$, that is zone size is 1 and we want correlation to be zero, then Walsh or Hadamard sequence sets satisfy the bound with equality.

Low Correlation Zone If $\delta = 1$, then it is a case of low correlation zone, hence we have

$$Ml_{cz} \leq N + 1.$$ 

When $M = 1$, $m$-sequences meet this bound. So, from the above results it is clear that product of zone size and number of sequences should match the period.
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Early Constructions

- Use $d$-form Functions: [Kim, Jang, No, Chung], 2005, Quaternary Constructions.
- $p$-ary sequences making making is $m$-sequences: [Jang, No, Chung, Tang 2007].
- Construction via Sub-field Reduction: [Gong, Golomb, Song, 2007].
Trace Sequences

Let $p$ be an odd prime, and $n = em$, $e$, $m$ are integers. Let $q = p^e$, For simplicity denote $\mathbb{GF}(p^e)$ as $\mathbb{GF}(q)$ and $\mathbb{GF}(p^n)$ as $\mathbb{GF}(q^m)$. Let $Tr^n_e(x)$ (and respectively $Tr^1_1(x)$ : $\mathbb{GF}(q^m) \rightarrow \mathbb{GF}(q)$.

The trace map is given by:

$$Tr^n_e(x) = \sum_{i=0}^{m-1} x^{p^{ei}}, \quad Tr^1_1(x) = \sum_{i=0}^{e-1} x^{p^i}.$$  

- Linear function.
- Random Function.
- Almost equal number of solutions: $Tr^n_e(x) = c$, for any $c \in \mathbb{GF}(q)$. 

Interleaved Sequences

Left shift operator $L$ on $a$ is defined as

$$L(a) = (a_1, \ldots, a_{N-1}, a_0).$$

- **A component sequence:** $a = (a_i, i = 0, 1, \ldots, N - 1)$ over $p$-ary symbols.
- **A shift Sequence:** $e = (e_j, j = 0, 1, \ldots, T - 1)$, where $e_j$ is an integer or $\infty$.
- **The Interleaved Sequence:** $N$ by $T$ matrix $A$ formed out of $a$ using the shift sequence $e$,

\[
A = \begin{pmatrix}
a_0 + e_0 & a_0 + e_1 & \cdots & a_0 + e_{T-1} \\
a_1 + e_0 & a_1 + e_1 & \cdots & a_1 + e_{T-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{N-1} + e_0 & a_{N-1} + e_1 & \cdots & a_{N-1} + e_{T-1}
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where the subscript addition is performed modulo $N$ and $a_i + \infty = 0$ for arbitrary $i$. 
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The $j$th column sequence $\{A\}_j$ of the matrix $A$ is

$$\{A\}_j = \begin{cases} 
0, & \text{if } e_j = \infty \\
L^{e_j}(a), & \text{else}
\end{cases}$$

By concatenating the successive rows of matrix $A$, an interleaved sequence $u$ of period $NT$ can be obtained as

$$u_{iT+j} = A_{i,j}, \quad 0 \leq i < N, \quad 0 \leq j < T.$$
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Cross Correlation

Let \( \underline{v} \) be a sequence interleaved by the \( p \)-ary component sequence \( \underline{b} = (b_i, \ i = 0, 1, \ldots, N - 1) \) and a shift sequence \( \underline{f} = (f_j, \ j = 0, 1, \ldots, T - 1) \)

\[
B = \begin{pmatrix}
    b_0 + f_0 & b_0 + f_1 & \cdots & b_0 + f_{T - 1} \\
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R_{u,v}(\tau) = \sum_{k=0}^{N \cdot T - 1} \omega^{u_k - v_{k+\tau}}, \omega = e^{j2\pi/p}.
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\]
We introduce two variables $\infty_1$ and $\infty_2$ and define $\infty - \infty = \infty$, $k - \infty = \infty_1$ and $\infty - k = \infty_2$ for any finite integer $k$.

Let $C_k(\tau)$ denote the cardinality of the set 
\[ \{ j | k = f_{j+\tau} - e_j \pmod{N}, \ 0 \leq j < T \} \]
for arbitrary $0 \leq k < N$ or $k = \infty$, $\infty_1$, or $\infty_2$, and

\[ I(a) = \sum_{i=0}^{N-1} \omega^{a_i} \quad \text{and} \quad I(b) = \sum_{i=0}^{N-1} \omega^{b_i}, \]

where $I(a)$ ($I(b)$ resp.) is called the imbalance of $a$ ($b$ resp.).
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$$I(a) = \sum_{i=0}^{N-1} \omega^{ai} \quad \text{and} \quad I(b) = \sum_{i=0}^{N-1} \omega^{bi},$$

where $I(a)$ ($I(b)$ resp.) is called the imbalance of $a$ ($b$ resp.).
m-sequence based Shift Sequence

- Let $\alpha$ be a primitive element over $\mathbb{GF}(p^n)$, and let $T = (p^n - 1)/(p^m - 1)$. We consider a shift sequence $e = (e_0, e_1, \ldots, e_{T-1})$ determined by

$$\alpha^{Te_t} = Tr^m_n(\alpha^t), \quad 0 \leq t < T.$$ 

- Let $f = e$.

$$R_{u,v}(\tau) = \begin{cases} 
p^{n-2m}(I(a) + 1)(I^*(b) + 1) - 1, & \text{if } \tau \not\equiv 0(\text{mod } T) \\
p^{n-m}R_{a,b}(d) + p^{n-m} - 1, & \text{if } \tau = d \cdot T \end{cases}.$$
A Construction with Balanced Sequences

If sequence $a$ or $b$ is balanced, i.e., $(I(a) + 1) = 0$ or $(I(b) + 1) = 0$, and $R_{a,b}(0) = 0$, then

$$R_{u,v}(\tau) = \begin{cases} p^{n-m}R_{a,b}(d) + p^{n-m} - 1, & \text{if } \tau = d \cdot T \text{ and } d \neq 0 \\ -1, & \text{Otherwise} \end{cases}$$

1. Let $U$ be a set of balance sequences of length $p^m - 1$ with the property that $R_{a,b}(0) = -1$ for any $a \neq b \in U$.

2. Now apply interleaved construction to all the sequences in $U$ based on the shift sequence $e$ resulting in a family $S$ of $p^m - 1$ sequences each of period $p^n - 1$. 
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2. Now apply interleaved construction to all the sequences in $U$ based on the shift sequence $e$ resulting in a family $S$ of $p^m - 1$ sequences each of period $p^n - 1$. 
An upper bound

The following upper bound follows from combinatorics of balanced $p$-ary sequences.

$$\left| U \right| \leq \min \left\{ \frac{(p^m-1)\left((p-1)p^{m-1}-1\right) \cdots (2p^{m-1}-1)}{p^m-1}, p^m-1 \right\}$$

$$= \begin{cases} 1, & m = 1 \\ 1, & p = 2, m = 2 \\ 5, & p = 2, m = 3 \\ p^m - 1, & \text{else} \end{cases}.$$
A Butson Hadamard Matrix $H_N$ of size $N$ over the set of complex $p^{th}$ roots of unity satisfying:

$$H_N H_N^T = N I_N,$$

where $T$ denotes conjugate transpose.

- $p = 2$: Binary Hadamard matrices.
- It is clear that, if you delete the first column of a Hadamard matrix, the rows satisfy the requirement for $U$. 
Connection to Hadamard Matrices

- A Butson Hadamard Matrix $H_N$ of size $N$ over the set of complex $p^{th}$ roots of unity satisfying:

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It is clear that, if you delete the first column of a Hadamard matrix, the rows satisfy the requirement for $U$. 
Non-Cyclic Hadamard Matrices

Let $H'_N$ be the punctured matrix of $H_N$ without the first column. A Hadamard matrix $H_N$ is said to be completely noncyclic if all the row vectors of the punctured matrix $H'_N$ are shift distinct.

- **In Small lengths: Binary Case:**

<table>
<thead>
<tr>
<th>length</th>
<th>Class</th>
<th>No. shift distinct row vectors in $H'_M$</th>
<th>Perm. which turns $H'_N$ to completely noncyclic</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>Class 1</td>
<td>2</td>
<td>None</td>
</tr>
<tr>
<td>8</td>
<td>Class 1</td>
<td>6</td>
<td>None</td>
</tr>
<tr>
<td>12</td>
<td>Class 1</td>
<td>11</td>
<td>(2, 3)</td>
</tr>
<tr>
<td>16</td>
<td>Class 1</td>
<td>15</td>
<td>(2, 3)</td>
</tr>
<tr>
<td>16</td>
<td>Class 2</td>
<td>15</td>
<td>(2, 3)</td>
</tr>
<tr>
<td>16</td>
<td>Class 3</td>
<td>15</td>
<td>(2, 3)</td>
</tr>
</tbody>
</table>

Table 1. Number of shift distinct vectors of known classes of $H'_N$, $N \leq 28$. 
A construction $K$ and a Mass Formula [Gong, Golomb, Song 2007].
A generalized sequence construction. A computational procedure which requires certain set of permutation functions.

- Paley Hadamard matrices: $q + 1$, $q = 3 \pmod{4}$, or $2(q + 1)$, $q = 1 \pmod{4}$, for any odd prime power $q$.
- Williamson Hadamard Construction: Powerful “plug-in” method to construct Hadamard matrices of order $4w$. 
Other works

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Simplex Code

- A recursive infinite construction.
  \[ 1 \rightarrow 0 \text{ and } -1 \rightarrow 1. \]

- Let, \(0_{2m-1}\): All zero vector of length \(2^m - 1\); and \(1_{2m-1}\): All one vector of length \(2^m - 1\).
- \(\Gamma_{2m-1}\): Vector with alternating 0 and 1's of length \(2^m - 1\).

Let \(\overline{a}\) be the binary complement of \(a\). Let \(S_m\) be a \((2^m - 1, 2^m)\) binary code consisting of \(2^m - 1\) nonzero codewords of length \(2^m - 1\) and \(0_{2m-1}\), which is recursively defined by

\[
S_m = \begin{pmatrix}
S_{m-1} & 0^T_{2m-1} \\
S_{m-1} & 1^T_{2m-1} \\
\end{pmatrix} \begin{pmatrix}
S_{m-1} \\
S_{m-1} \\
\end{pmatrix}, \quad m = 2, 3, \ldots
\]

(1)

where \(\overline{S_{m-1}}\) denotes the code whose \(i^{th}\) codeword is the complement of \(i^{th}\) one of \(S_{m-1}\).
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S_m = \begin{pmatrix}
S_{m-1} & 0_{2^m-1}^T \\
S_{m-1} & 1_{2^m-1}^T
\end{pmatrix}
\frac{S_{m-1}}{S_{m-1}}, \quad m = 2, 3, \ldots
\]  

(1)

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Example

- $S_1 = [0, 1]^T$.

<table>
<thead>
<tr>
<th>Index</th>
<th>Codewords</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>000</td>
</tr>
<tr>
<td>2</td>
<td>101</td>
</tr>
<tr>
<td>3</td>
<td>011</td>
</tr>
<tr>
<td>4</td>
<td>110</td>
</tr>
</tbody>
</table>

**Table:** Code words of $S_2$

- $(2^2 - 2 = 2)$ shift distinct codewords.
Example

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<td>2</td>
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</tr>
<tr>
<td>4</td>
<td>110</td>
</tr>
</tbody>
</table>

Table: Code words of $S_2$
Example Continued

<table>
<thead>
<tr>
<th>Index</th>
<th>Codewords</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>00000000</td>
</tr>
<tr>
<td>2</td>
<td>01100111</td>
</tr>
<tr>
<td>3</td>
<td>10101010</td>
</tr>
<tr>
<td>4</td>
<td>11001100</td>
</tr>
<tr>
<td>5</td>
<td>00011111</td>
</tr>
<tr>
<td>6</td>
<td>01111000</td>
</tr>
<tr>
<td>7</td>
<td>10110100</td>
</tr>
<tr>
<td>8</td>
<td>11010001</td>
</tr>
</tbody>
</table>

Table: Code words of $S_3$

Note that $(2, 4)$, $(3, 7)$, and $(5, 6)$, are shift equivalent codeword pairs. Thus we have five $(2^3 - 3 = 5)$ shift distinct codewords in $S_3$.

In general we show [Udaya-Tang05] $2^m \times (2^m - 1)$ punctured Sylvester Hadamard matrix has $2^m - m$ shift distinct row vectors.
Example Continued

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>0000000</td>
</tr>
<tr>
<td>2</td>
<td>0110011</td>
</tr>
<tr>
<td>3</td>
<td>1010101</td>
</tr>
<tr>
<td>4</td>
<td>1100110</td>
</tr>
<tr>
<td>5</td>
<td>0001111</td>
</tr>
<tr>
<td>6</td>
<td>0111100</td>
</tr>
<tr>
<td>7</td>
<td>1011010</td>
</tr>
<tr>
<td>8</td>
<td>1101001</td>
</tr>
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In general we show [Udaya-Tang05] $2^m \times (2^m - 1)$ punctured Sylvester Hadamard matrix has $2^m - m$ shift distinct row vectors.
Modified Simplex Code Construction

\[ R(a) = (a_0, a_1, \cdots, a_{2^m-2}). \]

Further let \( \psi_m \) be a code consisting of the zero codeword \( 0_{2^m-1} \) and \( 2^m - 1 \) nonzero codewords. We also use the operator \( R \) in straightforward manner to \( \psi_m \), i.e.,

\[ R(\psi_m) = \{ R(a), a \in \psi_m \}. \]

Then, the modified recursion is defined as

\[
\psi_{m+1} = \begin{pmatrix} \psi_m & 0^{T}_{2^m-1} & \psi_m \\ R(\psi_m) & 1^{T}_{2^m-1} & R(\psi_m) \end{pmatrix}.
\] (2)
Example

Let $\psi_4$ denote the code consisting of the following 16 codewords of length 15. It is easily checked that $\psi_4$ is corresponding to a punctured $16 \times 15$ Hadamard matrix.

<table>
<thead>
<tr>
<th>Index</th>
<th>Codewords</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0</td>
</tr>
<tr>
<td>2</td>
<td>0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1</td>
</tr>
<tr>
<td>3</td>
<td>0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1</td>
</tr>
<tr>
<td>4</td>
<td>0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1</td>
</tr>
<tr>
<td>5</td>
<td>0, 1, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 0, 1, 0, 0</td>
</tr>
<tr>
<td>6</td>
<td>0, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 0, 0, 1</td>
</tr>
<tr>
<td>7</td>
<td>0, 1, 1, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 1</td>
</tr>
<tr>
<td>8</td>
<td>0, 1, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1</td>
</tr>
<tr>
<td>9</td>
<td>1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0</td>
</tr>
<tr>
<td>10</td>
<td>1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 1, 1, 0</td>
</tr>
<tr>
<td>11</td>
<td>1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0</td>
</tr>
<tr>
<td>12</td>
<td>1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 1</td>
</tr>
<tr>
<td>13</td>
<td>1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0</td>
</tr>
<tr>
<td>14</td>
<td>1, 1, 0, 0, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1</td>
</tr>
<tr>
<td>15</td>
<td>1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0</td>
</tr>
<tr>
<td>16</td>
<td>1, 1, 1, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1</td>
</tr>
</tbody>
</table>
Generalized Hadamard matrices

- We introduced a general construction of infinite classes Generalized Hadamard matrices through Cocycles [Horadam, Udaya, 2002, 2003].
- Let $G$ be a group under addition. A **cocycle** on $G$ is a function $\psi : G \times G \rightarrow G$ satisfying $\psi(0, 0) = 0$ and

$$\psi(g, h) + \psi(g + h, k) = \psi(g, h + k) + \psi(h, k), \ \forall \ g, h, k \in G.$$  
- A useful tool in projective representation theory, cohomology of groups.
- We take $G$ to be the additive group of the finite field.
Multiplication Cocycles

The finite field multiplication function can be used to define cocycles leading to Hadamard matrix.

A cocyclic generalized Hadamard matrix looks as follows. Here $\alpha$ is a primitive element in $GF(p^m)$ and $N = p^m - 1$.

$$G = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & \alpha & \alpha^2 & \cdots & \alpha^{N-1} \\
0 & \alpha & \alpha^2 & \alpha^3 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \alpha^{N-1} & 1 & \alpha & \cdots & \alpha^{N-2}
\end{pmatrix},$$

Applying trace function to the above matrix leads to Binary Hadamard matrices.
Presemifield Multiple Cocycles

- The multiplication function of the above construction can be generalized to any pre-semifield. They lead to rich classes of Generalized Hadamard matrices.

- Recall that \((F, +, \ast)\) is a presemifield if \((F, +)\) is an abelian group (with additive identity 0), \((F \setminus \{0\}, \ast)\) is a quasigroup (that is, for any non-zero \(g, h \in F\) there are unique solutions in \(F\) to \(g \ast x = h\) and to \(y \ast g = h\)), and both distributive laws hold.

- A semifield is a presemifield with a multiplicative identity.

- If \((F, +, \ast)\) is a finite presemifield it has order \(q = p^a\) for some prime \(p\), its additive group \(G = (F, +)\) is the same as that of the finite field \(GF(q)\) of the same order.
**$Z_4$ construction**

[Elliot, Rao, 1982] give a general construction of fast transforms in the context of signal processing. We use one such construction to produce to new non-cyclic 4-ary Hadamard matrices.

\[ C_1 = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}, \quad C_2 = \begin{bmatrix} S_1 & S_1 \\ C_1 & -C_1 \end{bmatrix}, \]

\[ C_t = \begin{bmatrix} C_{t-1} & C_{t-1} \\ C_1 \otimes S_{t-2} & -C_1 \otimes S_{t-2} \end{bmatrix}, \quad t \geq 3, \]

where $S_t$ is a Sylvester Hadamard matrix of order $2^t$.

\[ S_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} S_1 & S_1 \\ S_1 & -S_1 \end{bmatrix}, \]
Example

\[ C_3 = \begin{bmatrix}
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
  1 & -i & -1 & i & 1 & -i & -1 & i \\
  1 & i & -1 & -i & 1 & i & -1 & -i \\
  1 & 1 & -i & -i & -1 & 1 & i & i \\
  1 & -1 & -i & i & -1 & 1 & i & -i \\
  1 & 1 & i & i & -1 & -1 & -i & -i \\
  1 & -1 & i & -i & -1 & 1 & -i & i \\
\end{bmatrix}, \]

\[ C_3 C_3^T = 8I_8. \]
Other Constructions

- Using two $m$-sequences and combining them in a relation similar to the recursion defined before [Jang,No,Chung,Tang07].
- The same method generalization to $q$-ary context.
- Quaternary matrix [Kim,No,Chung5].
Constructions with Flexible zone size

- New design of low-correlation zone sequence sets [Kim, Jang, No, Chung 2006], $\epsilon = 2$.
- Interleaved Sequence of Period $2N$ [Zhou, Tang, Gong, 2008], $\epsilon = 2$.
- Quaternary low correlation through Inverse Gray map, [Chung, Yang 2008].
- Some new designs in this workshop.
The above discussion can be extended to the case Frequency hopping CDMA communication systems.

Hamming Correlation as opposed to inner-product correlation defined above.

What does low correlation means in this context?
Conclusions

- Sequence design for Quasi-Synchronous Communication Systems.
- Interleaved Construction and connection to Non-cyclic Hadamard matrices.
- New designs with more choice of zone lengths and longer periods.