Low Correlation Zone Sequences over Finite Fields and Rings

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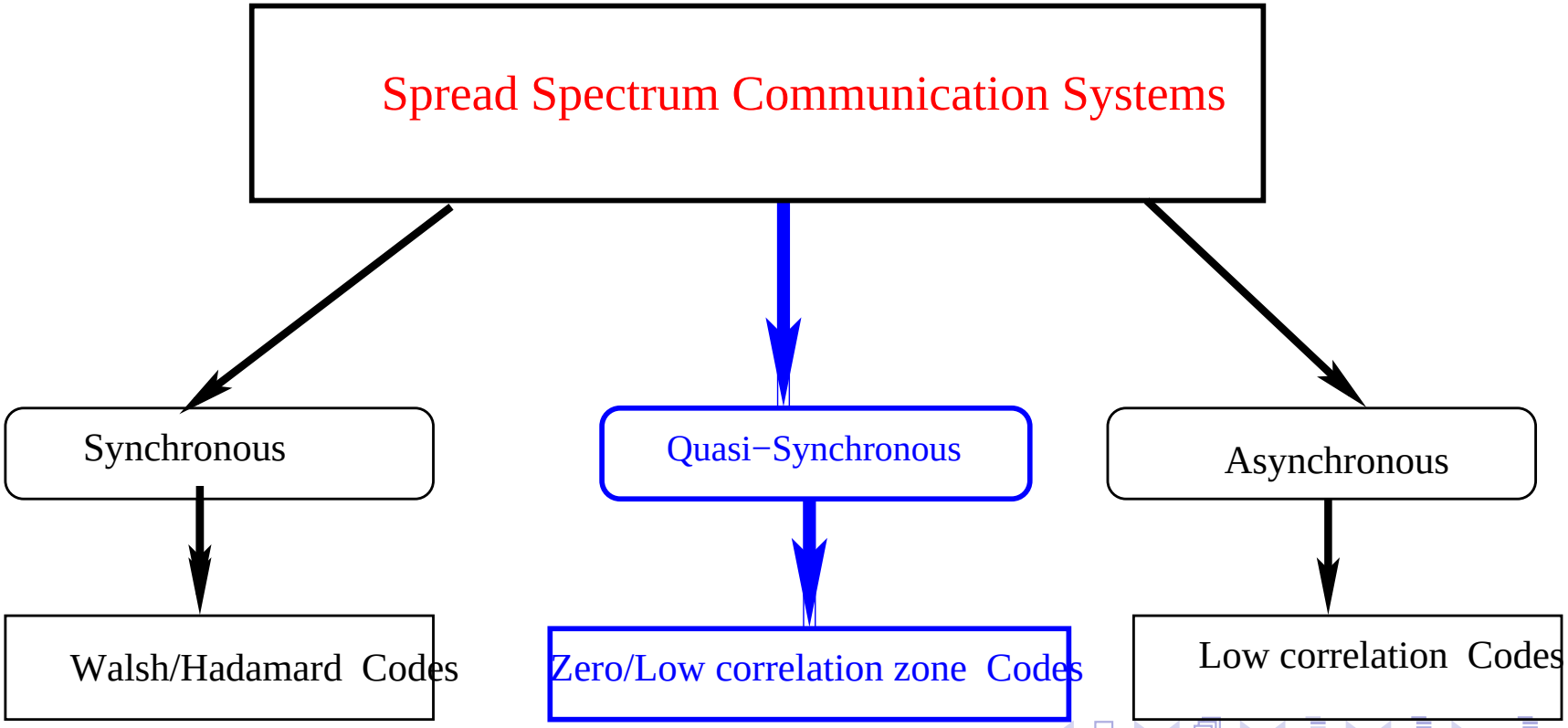
Context

Spread Spectrum Systems

- Sequences with low correlations are used as signature sequences in spread spectrum communication technologies.



Periodic Correlation Function

Definition

Given two complex sequences $a = \{a_0, a_1, \ldots, a_{N-1}\}$ and $b = \{b_0, b_1, \ldots, b_{N-1}\}$ of period N, we define the periodic cross-correlation between a and b as

$$R_{a,b}(\tau) = \sum_{i=0}^{N-1} a_i \, b_{i+\tau}^{\star}, 0 \leq \tau < N,$$

where $\star$ denotes for complex conjugation.

- A key measure to determine similarity between two sequences in the CDMA context.
- Helps in determining optimal sequences for synchronization, code acquisition and channel identification.

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Low Correlation Zone Requirement

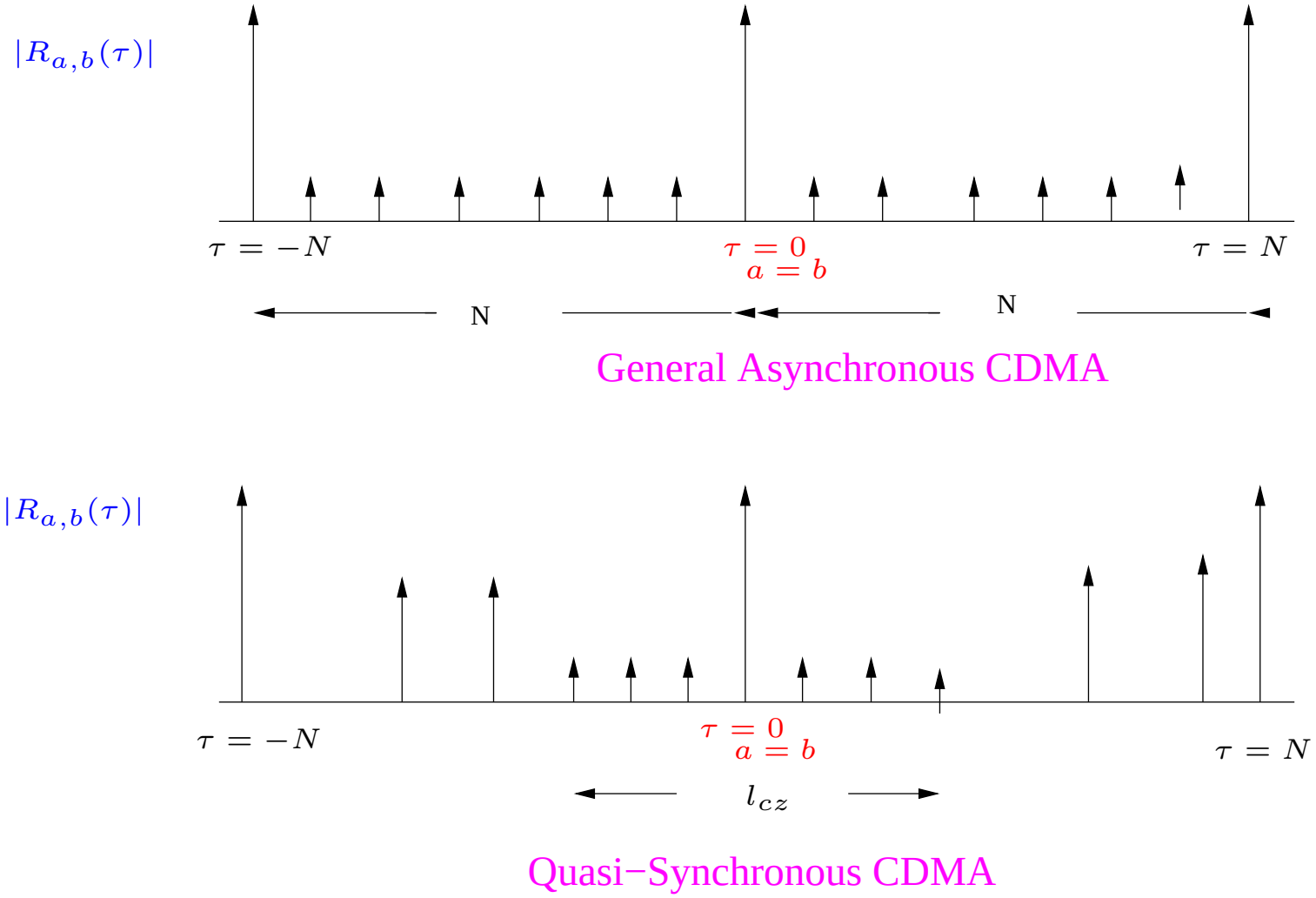

Figure: Correlation Zone

Orthogonality

Consider,

$$\{\mathcal{F} = \{a_0^r, a_1^r, \ldots, a_{N-1}^r\}; r = 1 \cdots M\},$$

a family of M sequences of each length N.
We say the family is **Orthogonal**, if:

$$R_{r,s}(\tau) = \sum_{i=0}^{N-1} a_i^r a_{i+\tau}^{s\star}, \left\{ \begin{array}{l} = N, \text{ for } \tau = 0, r = s \\ = 0, \text{ for } \tau = 0, r \neq s \end{array} \right\}$$

$\star$: Complex Conjugate Note that $R_{r,s}(\tau)$ is zero only for $\tau = 0$.

Walsh sequence set or vectors of Hadamard matrices satisfy the above criterion. They are widely used in synchronous CDMA communication systems.

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For a small number ϵ $(\epsilon < N)$ and a zone l_{cz}, we say the family is **Generalized Quasi-Orthogonal** $(GQO(N, M, \epsilon, l_{cz}))$, if :

$$R_{r,s}(\tau) = \sum_{i=0}^{N-1} a_i^r a_{i+\tau}^{s\star}, \left\{ \begin{array}{l} = N, \text{ for } \tau = 0, r = s \\ = \epsilon, \text{ for } \tau = 0, r \neq s \\ = \epsilon, \text{ for } 0 < |\tau| \leq l_{cz} \end{array} \right\}$$

A requirement is that maximum non-trivial correlation amongst members of a GQO should be a small quantity.

Sequences satisfying this property $l_{cz} = 0$ are studied by [Yang,Kim,Kumar,2000]: Quasi-orthogonal CDMA systems, IEEE-IT, 2000.

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Polyphase sequences

- The above definitions are for any complex valued sequences. In practice the signature sequences in CDMA are generally defined over points on unit circle.
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Aperiodic Correlation requirements

Consider $GQO(N, M, \epsilon, l_{cz})$. M sequences of length M, with low correlation zone l_{cz} and low correlation value ϵ.
We also sometimes need aperiodic correlations of sequences in ($GQO(N, M, \epsilon, l_{cz})$) satisfy the following property:

$$\delta_{r,s}(\tau) = \sum_{i=0}^{N-\tau} \omega^{a_i^r - a_{i+\tau}^s}, \left\{ \begin{array}{ll} = N, & \text{for } \tau = 0, r = s \\ \leq \epsilon, & \text{for } \tau = 0, r \neq s \\ \leq \epsilon, & \text{for } 0 < |\tau| \leq l_{cz} \end{array} \right\}.$$

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Theoretical Limits

Bounds on correlations for Generalized Quasi-Orthogonal sequences

- Welch Bound on Correlations: For a family M periodic complex sequences of length N, the periodic correlations satisfy

$$R(\tau) \geq \sqrt{N}.$$

- Sidelnikov Bound: For the binary case,

$$R(\tau) \geq \sqrt{2N}.$$

- Several variations of the above bounds exist. But they all assume that l_{cz} is equal to length of the sequences. Hence they do not provide sufficient hint for optimal constructions.

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Welch Boound on Periodic Sequences

Consider set of M vectors of length N.

$$\{(a_1^s, a_2^s, \cdots a_N^s), 0 \leq s < M\}, \text{ each of unit norm}$$

$$R_{s,s}(0) = \sum_{i=0}^{N-1} |a_i^s|^2 = 1.$$

Let δ be the non-trivial value of inner product between any two vectors in the matrix:

$$\text{Inner Product } \delta(r,s) = R_{r,s}(0) = \sum_{i=0}^{N-1} a_i^r (a_i^s)^\star.$$

$$\delta^{2k} = R_{s,t}(0)^{2k} \geq \frac{1}{(M-1)} \left[\frac{M}{\binom{N+k-1}{k}} - 1 \right]$$

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Bound for Generalized Quasi-Orthogonal Sequences

Construct a new matrix H from M vectors of the previous slide: Include all vectors as rows of H and further add $1, \cdots l_{cz}$ shifts of these vectors to H.

$$H(iN + j, t] := a^i_{t+j},$$

where $0 \leq i \leq M - 1$, $0 \leq j \leq l_{cz} - 1$ and $0 \leq t < N$.
Clearly there are Ml_{cz} vectors in H; Using Inner-product theorem

$$\delta^{2k} \geq \frac{1}{(Ml_{cz} - 1)} \Big[\frac{Ml_{cz}}{\binom{N+k-1}{k}} - 1 \Big].$$

Because of $R_{s,t}(\tau) = R_{t,s}(-\tau)^\star$,

$$R_{s,t}(\tau) \geq \frac{1}{(Ml_{cz} - 1)} \Big[\frac{Ml_{cz}}{\binom{N+k-1}{k}} - 1 \Big], 0 \leq |\tau| \leq l_{cz}.$$

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Special Cases

- Welch's result assumes that norm of the vectors is unity, i.e, $R_{r,r}(0) = \sum_{i=0}^{N-1} |a_i^r|^2 = 1, 0 \leq r < M$.
- δ^2 is the value of non-trivial correlation between any two sequences with shifts in the range $0 \leq \tau < l_{cz}$.
  In practice, the sequences are over unit circle (polyphase or biphase), then $R_{r,r}(0) = N$.
- Choose $k = 1$ in the previous result and apply the previous theorem by normalizing sequences, we get

$$\delta^2 \geq N \frac{(Ml_{cz} - N)}{Ml_{cz} - 1}.$$

- Rearranging, we get

$$(Ml_{cz} - 1)(1 - \delta^2/N) \leq N - 1.$$

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Zero/low correlation zone constraints

Zero Correlation Zone If $\delta^2/N = 0$, then we have,

$$Ml_{cz} \leq N.$$

If $l_{cz} = 1$, that is zone size is 1 and we want correlation to be zero, then Walsh or Hadamard sequence sets satisfy the bound with equality.

Low Correlation Zone If $\delta = 1$, then it is a case of low correlation zone, hence we have

$$Ml_{cz} \leq N + 1.$$

When $M = 1$, m-sequences meet this bound. So, from the above results it is clear that product of zone size and number of sequences should match the period.

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Early Constructions

- The first Construction: Binary sequences using PN and GMW constructions: [Lin,Chang,1997], [Long,Zhang,Hu 1998], [TangFan2000].
- Use d-form Functions: [Kim, Jang, No,Chung], 2005, Quaternary Constructions.
- Interleaved constructions (Optimal and nearly Optimal): [Tang,Udaya 2005] Using Hadamard Matrices.
- p-ary sequences making making is m-sequences: [Jang, No, Chung, Tang 2007].
- Construction via Sub-field Reduction: [Gong,Golomb,Song, 2007].

Trace Sequences

Let p be an odd prime, and $n = em$, e, m are integers. Let $q = p^e$, For simplicity denote $\mathbf{GF}(p^e)$ as $\mathbf{GF}(q)$ and $\mathbf{GF}(p^n)$ as $\mathbf{GF}(q^m)$.
Let $Tr_e^n(x)$ (and respectively

$$Tr_1^e(x) : \mathbf{GF}(q^m) \to \mathbf{GF}(q).$$

The trace map is given by:

$$Tr_e^n(x) = \sum_{i=0}^{m-1} x^{p^{ei}}, \quad Tr_1^e(x) = \sum_{i=0}^{e-1} x^{p^i}.$$

- Linear function.
- Random Function.
- Almost equal number of solutions: $Tr_e^n(x) = c$, for any $c \in \mathbf{GF}(q)$.

Interleaved Sequences

Left shift operator L on a is defined as
$L(a) = (a_1, \ldots, a_{N-1}, a_0)$.

- A component sequence: $\underline{a} = (a_i,\ i = 0, 1, \ldots, N-1)$ over p-ary symbols.
- A shift Sequence: $\underline{e} = (e_j,\ j = 0, 1, \ldots, T-1)$, where e_j is an integer or ∞.
- The Interleaved Sequence: N by T matrix A formed out of a using the shift sequence $\underline{e}$,

$$A = \begin{pmatrix} a_{0+e_0} & a_{0+e_1} & \cdots & a_{0+e_{T-1}} \\ a_{1+e_0} & a_{1+e_1} & \cdots & a_{1+e_{T-1}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1+e_0} & a_{N-1+e_1} & \cdots & a_{N-1+e_{T-1}} \end{pmatrix},$$

where the subscript addition is performed modulo N and $a_{i+\infty} = 0$ for arbitrary i.

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- The jth column sequence $\{A\}_j$ of the matrix A is

$$\{A\}_j = \begin{cases} 0, & \text{if } e_j = \infty \\ L^{e_j}(\underline{a}), & \text{else} \end{cases},$$

- By concatenating the successive rows of matrix A, an interleaved sequence $\underline{u}$ of period NT can be obtained as

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Cross Correlation

- Let $\underline{v}$ be a sequence interleaved by the p-ary component sequence $\underline{b} = (b_i,\, i = 0, 1, \ldots, N-1)$ and a shift sequence $\underline{f} = (f_j,\, j = 0, 1, \ldots, T-1)$

$$B = \begin{pmatrix} b_{0+f_0} & b_{0+f_1} & \cdots & b_{0+f_{T-1}} \\ b_{1+f_0} & b_{1+f_1} & \cdots & b_{1+f_{T-1}} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N-1+f_0} & b_{N-1+f_1} & \cdots & b_{N-1+f_{T-1}} \end{pmatrix}.$$

$$R_{\underline{u},\underline{v}}(\tau) = \sum_{k=0}^{N\cdot T-1} \omega^{u_k - v_{k+\tau}}, \omega = e^{j2\pi/p}.$$

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We introduce two variables ∞_1 and ∞_2 and define $\infty - \infty = \infty$, $k - \infty = \infty_1$ and $\infty - k = \infty_2$ for any finite integer k.

Let $C_k(\tau)$ denote the cardinality of the set

$\{j | k = f_{j+\tau} - e_j \pmod N,\ 0 \leq j < T\}$

for arbitrary $0 \leq k < N$ or $k = \infty$, ∞_1, or ∞_2, and

$$I(\underline{a}) = \sum_{i=0}^{N-1} \omega^{a_i} \text{ and } I(\underline{b}) = \sum_{i=0}^{N-1} \omega^{b_i},$$

where $I(\underline{a})$ ($I(\underline{b})$ resp.) is called the imbalance of $\underline{a}$ ($\underline{b}$ resp.).

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for arbitrary $0 \leq k < N$ or $k = \infty$, ∞_1, or ∞_2, and

$$I(\underline{a}) = \sum_{i=0}^{N-1} \omega^{a_i} \text{ and } I(\underline{b}) = \sum_{i=0}^{N-1} \omega^{b_i},$$

where $I(\underline{a})$ ($I(\underline{b})$ resp.) is called the imbalance of $\underline{a}$ ($\underline{b}$ resp.).

m-sequence based Shift Sequence

- Let α be a primitive element over $\mathbf{GF}(p^n)$, and let $T = (p^n - 1)/(p^m - 1)$. We consider a shift sequence $\underline{e} = (e_0, e_1, \ldots, e_{T-1})$ determined by

$$\alpha^{Te_t} = Tr_m^n(\alpha^t), 0 \leq t < T.$$

- Let $\underline{f} = \underline{e}$.
-

$$R_{\underline{u},\underline{v}}(\tau) = \begin{cases} p^{n-2m}(I(\underline{a}) + 1)(I^*(\underline{b}) + 1) - 1, \\ \text{if } \tau \not\equiv 0 (\bmod T) \\ \\ p^{n-m} R_{\underline{a},\underline{b}}(d) + p^{n-m} - 1, \\ \text{if } \tau = d \cdot T \end{cases}.$$

A Construction with Balanced Sequences

If sequence $\underline{a}$ or $\underline{b}$ is balanced, i.e., $(I(\underline{a})+1)=0$ or $(I(\underline{b})+1)=0$, and $R_{\underline{a},\underline{b}}(0)=0$, then

$$R_{\underline{u},\underline{v}}(\tau)=\begin{cases} p^{n-m}R_{\underline{a},\underline{b}}(d)+p^{n-m}-1, & \text{if } \tau=d\cdot T \text{ and } d\neq 0 \\ -1, & \text{Otherwise} \end{cases}.$$

1. Let U be a set of balance sequences of length p^m-1 with the property that $R_{\underline{a},\underline{b}}(0)=-1$ for any $a\neq b\in U$.
2. Now apply interleaved construction to all the sequences in U based on the shift sequence $\underline{e}$ resulting in a family $\mathcal{S}$ of p^m-1 sequences each of period p^n-1.

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An upper bound

The following upper bound follows from combinatorics of balanced p-ary sequences.

$$
\begin{aligned}
|U| &\leq \min\left\{ \frac{\binom{p^m-1}{p^{m-1}}\binom{(p-1)p^{m-1}-1}{p^{m-1}}\cdots\binom{2p^{m-1}-1}{p^{m-1}}}{p^m-1}, \right. \\
&\qquad \left. p^m-1 \right\} \\
&= \begin{cases} 1, & m=1 \\ 1, & p=2, m=2 \\ 5, & p=2, m=3 \\ p^m-1, & \text{else} \end{cases}.
\end{aligned}
$$

Connection to Hadamard Matrices

- A Butson Hadamard Matrix H_N of size N over the set of complex p^{th} roots of unity satisfying:

$$H_N H_N^T = N I_N,$$

  where T denotes conjugate transpose.
- $p = 2$: Binary Hadamard matrices.
- It is clear that, if you delete the first column of a Hadamard matrix, the rows satisfy the requirement for U.

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Non-Cyclic Hadamard Matrices

Let H'_N be the punctured matrix of H_N without the first column. A Hadmard matrix H_N is said to be completely noncyclic if all the row vectors of the punctured matrix H'_N are shift distinct.

- In Small lengths: Binary Case:

length	Class	No. shift distinct row vectors in H'_M	Perm. which turns H'_N to completely noncyclic
4	$Class\ 1$	2	$None$
8	$Class\ 1$	6	$None$
12	$Class\ 1$	11	$(2,3)$
16	$Class\ 1$	15	$(2,3)$
16	$Class\ 2$	15	$(2,3)$
16	$Class\ 3$	15	$(2,3)$

Table 1. Number of shift distinct vectors of known classes of $H'_N, N \leq 28$.

Other works

- A construction K and a Mass Formula[Gong,Golomb,Song 2007]. A generalized sequence construction. A computational procedure which requires certain set of permutation functions.
- Paley Hadamard matrices: $q+1$, $q=3(\bmod 4)$, or $2(q+1)$, $q=1(\bmod 4)$, for any odd prime power q.
- Williamson Hadamard Construction: Powerful "plug-in" method to construct Hadamard matrices of order $4w$.

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Simplex Code

- A recursive infinite construction.
  $1 \rightarrow 0$ and $-1 \rightarrow 1$.
- Let, $\mathbf{0}_{2^m-1}$: All zero vector of length $2^m - 1$;
  and $\mathbf{1}_{2^m-1}$: All one vector of length $2^m - 1$.
  $\mathbf{\Gamma}_{2^m-1}$: Vector with alternating 0 and 1ús of length $2^m - 1$.
  Let $\overline{a}$ be the binary complement of a. Let $\mathcal{S}_m$ be a $(2^m - 1, 2^m)$ binary code consisting of $2^m - 1$ nonzero codewords of length $2^m - 1$ and $\mathbf{0}_{2^m-1}$, which is recursively defined by

$$\mathcal{S}_m = \begin{pmatrix} \mathcal{S}_{m-1} & \mathbf{0}^T_{2^{m-1}} & \mathcal{S}_{m-1} \\ \mathcal{S}_{m-1} & \mathbf{1}^T_{2^{m-1}} & \overline{\mathcal{S}_{m-1}} \end{pmatrix}, \; m = 2, 3, \cdots . \tag{1}$$

  where $\overline{\mathcal{S}_{m-1}}$ denotes the code whose i^{th} codeword is the complement of ith one of $\mathcal{S}_{m-1}$.

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Example

- $\mathcal{S}_1 = [0, 1]^T$.
-

Index	Codewords
1	000
2	101
3	011
4	110

Table: Code words of $\mathcal{S}_2$

- $(2^2 - 2 = 2)$ shift distinct codewords.

Example

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-

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Table: Code words of $\mathcal{S}_2$

- $(2^2 - 2 = 2)$ shift distinct codewords.

Example Continued

Index	Codewords
1	$000\mathbf{0}000$
2	$011\mathbf{0}011$
3	$101\mathbf{0}101$
4	$110\mathbf{0}110$
5	$000\mathbf{1}111$
6	$011\mathbf{1}100$
7	$101\mathbf{1}010$
8	$110\mathbf{1}001$

Table: Code words of $\mathcal{S}_3$

- Note that $(2, 4)$, $(3, 7)$, and $5, 6)$, are shift equivalent codeword pairs. Thus we have five $(2^3 - 3 = 5)$ shift distinct codewords in $\mathcal{S}_3$.
  In general we show [Udaya-Tang05] $2^m \times (2^m - 1)$ punctured Sylvester Hadamard matrix has $2^m - m$ shift distinct row vectors.

Example Continued

Index	Codewords
1	$000\mathbf{0}000$
2	$011\mathbf{0}011$
3	$101\mathbf{0}101$
4	$110\mathbf{0}110$
5	$000\mathbf{1}111$
6	$011\mathbf{1}100$
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Modified Simplex Code Construction

$$R(a) = (\overline{a_0}, a_1, \cdots, a_{2^m-2}).$$

Further let ψ_m be a code consisting of the zero codeword $\mathbf{0}_{2^m-1}$ and $2^m - 1$ nonzero codewords. We also use the operator R in straightforward manner to ψ_m, i.e.,

$$R(\psi_m) = \{R(a), a \in \psi_m\}.$$

Then, the modified recursion is defined as

$$\psi_{m+1} = \begin{pmatrix} \psi_m & \mathbf{0}^T_{2^{m-1}} & \psi_m \\ R(\psi_m) & \mathbf{1}^T_{2^{m-1}} & \overline{R(\psi_m)} \end{pmatrix}. \tag{2}$$

Starter matrix

Example

Let ψ_4 denote the code consisting of the bellowing 16 codewords of length 15. It is easily checked that ψ_4 is corresponding to a punctured 16×15 Hadamard matrix.

Index	Codewords
1	$0,0,0,0,0,0,0,0,0,0,0,0,0,0,0$
2	$0,0,0,0,0,0,0,1,1,1,1,1,1,1,1$
3	$0,0,0,1,1,1,1,0,0,0,0,1,1,1,1$
4	$0,0,1,0,1,1,1,0,1,1,1,0,0,0,1$
5	$0,1,0,1,0,1,1,1,0,1,1,0,0,1,0$
6	$0,1,1,0,0,1,1,1,1,0,0,1,1,0,0$
7	$0,1,1,1,1,0,0,0,1,0,1,0,1,1,0$
8	$0,1,1,1,1,0,0,1,0,1,0,1,0,0,1$
9	$1,0,0,1,1,0,1,1,1,0,1,1,0,0,0$
10	$1,0,1,0,1,0,1,1,0,1,0,0,1,1,0$
11	$1,0,1,1,0,1,0,0,0,1,1,1,1,0,0$
12	$1,0,1,1,0,1,0,1,1,0,0,0,0,1,1$
13	$1,1,0,0,1,1,0,0,1,1,0,1,0,1,0$
14	$1,1,0,0,1,1,0,1,0,0,1,0,1,0,1$
15	$1,1,0,1,0,0,1,0,1,1,0,0,1,0,1$
16	$1,1,1,0,0,0,1,0,0,0,1,1,0,1,1$

Generalized Hadamard matrices

- We introduced a general construction of infinite classes Generalized Hadamard matrices through Cocycles [Horadam, Udaya,2002,2003] .
- Let G be a group under addition. A *cocycle* on G is a function $\psi : G \times G \rightarrow G$ satisfying $\psi(0,0) = 0$ and

$$\psi(g,h) + \psi(g+h,k) = \psi(g,h+k) + \psi(h,k), \ \forall \ g,h,k \in G.$$

- A useful tool in projective representation theory, cohomology of groups.
- We take G to be the additive group of the finite field.

Multiplication Cocycles

The finite field multiplication function can be used to define cocycles leading to Hadamard matrix.
A cocyclic generalized Hadamard matrix looks as follows. Here α is a primiive element in $GF(p^m)$ and $N = p^m - 1$.

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \alpha & \alpha^2 & \cdots & \alpha^{N-1} \\ 0 & \alpha & \alpha^2 & \alpha^3 & \cdots & 1 \\ . & . & . & . & \cdots & . \\ 0 & \alpha^{N-1} & 1 & \alpha & \cdots & \alpha^{N-2} \end{pmatrix},$$

Applying trace function to the above matrix leads to Binary Hadamad matrices.

Presemifield Multiple Cocycles

- The multiplication function of the above construction can be generalized to any pre-semifield. They lead to rich classes of Generalized Hadamard matrices.
- Recall that $(F, +, *)$ is a *presemifield* if $(F, +)$ is an abelian group (with additive identity 0), $(F \backslash \{0\}, *)$ is a quasigroup (that is, for any non-zero $g, h \in F$ there are unique solutions in F to $g * x = h$ and to $y * g = h$), and both distributive laws hold.
- A *semifield* is a presemifield with a multiplicative identity.
- If $(F, +, *)$ is a finite presemifield it has order $q = p^a$ for some prime p, its additive group $G = (F, +)$ is the same as that of the finite field $\mathrm{GF}(q)$ of the same order.

Z_4 construction

[Elliot, Rao, 1982] give a general construction of fast transforms in the context of signal processing. We use one such construction to produce to new non-cyclic 4-ary Hadamard matrices.

$$C_1 = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}, C_2 = \begin{bmatrix} S_1 & S_1 \\ C_1 & -C_1 \end{bmatrix},$$

$$C_t = \begin{bmatrix} C_{t-1} & C_{t-1} \\ C_1 \otimes S_{t-2} & -C_1 \otimes S_{t-2} \end{bmatrix}, t \geq 3,$$

where S_t is a Sylvester Hadmard matrix of order 2^t.

$$S_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, C_2 = \begin{bmatrix} S_1 & S_1 \\ S_1 & -S_1 \end{bmatrix},$$

Example

$$C_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & 1 & -i & -i & -1 & -1 & i & i \\ 1 & -1 & -i & i & -1 & 1 & i & -i \\ 1 & 1 & i & i & -1 & -1 & -i & -i \\ 1 & -1 & i & -i & -1 & 1 & -i & i \end{bmatrix},$$

$$C_3 C_3^T = 8 I_8$$

.

Other Constructions

- Using two m-sequences and combining them in a relation similar to the recursion defined before [Jang,No,Chung,Tang07].
- The same method generalization to q-ary context.
- Quaternary matrix [Kim,No,Chung5].

Constructions with Flexible zone size

- New design of low-correlation zone sequence sets [Kim,Jang,No,Chung 2006], $\epsilon = 2$.
- Interleaved Sequence of Period $2N$ [Zhou,Tang,Gong,2008],$\epsilon = 2$
- Quaternary low correlation through Inverse Gray map, [Chung,Yang2008].
- Some new designs in this workshop.

Frequency Hopping

- The above discussion can be extended to the case Frequency hopping CDMA communication systems.
- Hamming Correlation as opposed to inner-product correlation defined above.
- What does low correlation means in this context?

Conclusions

- Sequence design for Quasi-Synchronous Communication Systems.
- Interleaved Construction and connection to Non-cyclic Hadamard matrices.
- New designs with more choice of zone lengths and longer periods.